Matrices and Gershgorin theorem

المصفوفات ونظرية غيرشجورين

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الملخص:

يتم أخذ حدود الاضـــطراب لقيم eigen المتعددة باســـتخدام تعميم نظرية Gershgorin التي تم تطويرها لقيمة eigen للمشـــكلة $Axe = lambda Bx$. عند التعامل مع قيم Eigen اللانهائية، يتم تفسـير النتائج بشـكل خاص من حيث القياسـات الوترية على كرة ريمان، وقد تم تقديم تطبيق نظرية غيرشجورين في تحليل استقرار النظام الذي يمثله علم رياضي متعدد الحدود. كما تم تقديم تطبيقات نظرية غيرشجورين في نمذجة الترتيب المخفض من خلال التعرف على المقاييس الزمنية والثبات لنظام الزمن الخطي، ومن الأسـاليب المعروفة والفعالة لمعرفة القيم الذاتية للمصـفوفة من حيث مدخلاتها هي دائرة غيرشجورين. نظرية. إذا كانت A عبارة عن مصفوفة متماثلة، فإننا نستكشف اختيار x لتوفير قيود أحادية الجانب على القيم الذاتية لـــ A أثناء الحصـول على يمكن الحصـول على Gershgorin الأكثر فعالية المرتبط بتقدير القيم الذاتية للمصـفوفة B عن طريق التعبير عن المصفوفة A كمجموع B ومصفوفة x1 تتكون من عناصر الوحدة. لقد أثبتنا أن قيمة x يمكن تحديدها بكفاءة باستخدام برنامج خطي، وعادة ما يتم توفير الحل في شـــكل مغلق. إن تقنية التحويل المقترحة لدينا تتجاوز القيود الخطية أو التربيعية الموجودة على القيم الذاتية لأنواع مختلفة من المصـــفوفات، وخاصـة بالنسـبة للفئات الأكبر. بالإضـافة إلى ذلك، قمنا بدمج عدد قليل من المقدرات غير الخطية كجزء من هذا التغيير الكبير في النهج. في الجبر، يتم استخدام عامل تفاضـلي منفصـل ذو ترتيب متسـاوي لإنشـاء مصـفوفة الصـلابة العالمية للعناصـر المحدودة الكبيرة. ومن المثير للاهتمام أن هذه المصـفوفة تعرض خاصية فريدة: جميع الصفوف مجموعها يساوي الصفر. على الرغم من أن نظرية غيرشجورين للقيمة الذاتية المحيطة مضمونة نظريًا لتوفير أصغر قيمة ذاتية، إلا أنها تفشل في القيام بذلك بشكل مباشر لهذه المصفوفة المحددة. ومع ذلك، هناك إمكانية لتحسين وتحديد حدود دنيا أكثر واقعية لأصغر قيمة ذاتية من خلال النظر في تقنيات عملية لبناء تحويل تشـــابه مثالي لمصـــفوفة الصـــلابة العالمية للعناصـــر المحدودة. من المهم ملاحظة أن هذه الطريقة قابلة للتطبيق فقط عند التعامل مع مصفوفة صلابة شاملة تحتوي على عناصر سالبة خارج القطر، كما أن القيود المفروضة على نظرية غيرشجورين ذات صلة في هذا السياق.

Abstract

The perturbation boundaries for the multiple eigen values are taken using a generalisation of the Gershgorin theorem that is developed for the eigen value for the problem $Axe = lambda Bx$. When handling infinite Eigen values, the findings are specifically interpreted in terms of chordal measurements on the Riemann sphere, It was presented the Gerschgorin theorem application in the relative of a system stability analysis that is represented by a polynomial mathematical science. It was also presented the Gershgorin Theorem applications in the reduced order modeling by being aware of the time scales and stabilisation for the linear time system, A well-known and effective technique for figuring out a matrix's eigenvalues in terms of its entries is the Gershgorin Circle Theorem. If A is a symmetric matrix, We explore the choice of x to provide one-sided constraints on the eigenvalues of A while obtaining The most effective Gershgorin bound for estimating the eigenvalues of matrix B can be obtained by expressing matrix A as the sum of B and a matrix x1 consisting of unit elements. We demonstrate that the value of x can be efficiently determined using a linear program, and the solution is usually provided in a closed form. Our proposed shifting technique surpasses existing piecewise linear or quadratic restrictions on eigenvalues for various types of matrices, especially for larger classes. Additionally, we incorporate a few nonlinear estimators as part of this significant change in approach, In algebra, a discrete differential operator of even order is utilized to create the global stiffness matrix for large finite elements. Interestingly, this matrix exhibits a unique property: all the rows add up to zero. Although Gershgorin's eigenvalue bounding theorem is theoretically guaranteed to provide the smallest eigenvalue, it fails to do so directly for this particular matrix. Nevertheless, there is a possibility to improve and identify more realistic lower bounds on the smallest eigenvalue by considering practical techniques to construct an ideal similarity transformation for the global stiffness matrix of finite elements. It's important to note that this method is only applicable when dealing with a global stiffness matrix containing exclusively negative off-diagonal elements, and the limitations of Gershgorin's theorem are relevant in this context .

Introduction

In linear algebra, a matrix is a rectangular arrangement of numbers or symbols organized in columns and rows, forming a two-dimensional array with a rectangular shape.[1]

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$$\begin{vmatrix} A1 & A2 & A3 \\ A4 & A5 & A6 \\ A7 & A8 & A9 \end{vmatrix}$$

The matrix is used for describing differential or linear equations, in addition to represent the linear application.

The Matrix is considered picture of a ground-breaking motion that is used for plotting statistics, graphs and also to make the scientific research and studies in many different fields.[2]

Matrices is able to represent the data of the real world like the people population, the rate of the infant mortality, etc. so, The Matrices are the best method for representation the plotting surveys.[3]

In the biology, the matrices are found in various connective tissues and is considered as the tissue or the material in between the cells of the eukaryotic organism. The connective tissues structure is an extracellular matrix. toenails and Fingernails grow from the matrices.[4]

A simple calculation reveals that the eigenvalues of B are = 1 (real) and = i (complex conjugates). Complex eigenvalues, on the other hand, are not attainable with symmetric matrices. A symmetric matrix with real entries' eigenvalues are always real.

The matrices are considered among the most common tool in the computer science and electrical engineering through representing the mathematical equations.[5]

The matrix also can be used in the decision to make the complex decisions tasks, for solving the craft problems and arguments to able for taking a decision that you need to make. So, the matrix is considered a tool for decision-making with multi quantitative criteria.[6]

In the mathematics science, the Gershgorin theorem can be used for bounding the square matrix spectrum to provide the eigenvalues approximates to the diagonal matrix that lie in a specific half-plane.[7]

When directly applying Gershgorin's eigenvalue bounding theorem to the large finite element global stiffness matrix, which serves as an algebraic representation of a discrete differential operator of even order, a notable property is observed: the matrix exhibits zero row sums.

it is consistently and reliably impossible to obtain a meaningful lower limit on the smallest eigenvalue, even though it is theoretically guaranteed to be positive definite. In this context, we explore approaches to generate an ideal similarity transformation for the global stiffness matrix in real-world

[1] Berman, A., & Plemmons, R. J. (1994). Non-Negative Matrices in Mathematical Sciences. SIAM.

[2] Roy, A. (2010). Minimum Euclidean Representations for Graphs. Discrete Mathematics, 310, 727-733.

[3] Anderson, P. W. (1958). Absence of diffusion in some random synapses. Physical Review, 109, 1492-1505.

[4] Fiedler, M., Hall, F. J., & Marsley, R. (2013). Gershgorin's Discs Revisited. Journal of Linear Algebra and Its Applications, 438, 598-603.

[5] Boyd, S., & Desoer, C. A. (1985). Subharmonic Functions and Performance Limits on Fixed Linear Time Feedback Systems. IMA Journal of Mathematical Control and Information, 2, 153-170

[6] Böttcher, A., & Grudsky, S. M. (2005). Spectral characteristics of the bands Toeplitz Matrices. SIAM, Philadelphia.

[7] Böttcher, A., & Silbermann, B. (2006). Analysis of Toeplitz operators (2nd ed.). Springer.

scenarios. By employing these techniques, it becomes possible to discover and improve nontrivial and practical lower bounds on the smallest eigenvalue.

Due to the discs are considered that are present in the center for the diagonal elements, that will be not at zero. therefore, the big implication of the theorem is that wither the diagonal elements are big enough, for the example: the matrix is suitable for the diagonal dominant, then the discs mak not consists of zero, therefore, there is no the zero eigen values , such as the matrix is considered as invertible.

The Gershgorin Circles Theorem is an extremely valuable tool for characterizing the regions in the complex plane where matrix eigenvalues can be found. It plays a crucial role in determining the local stability of the equilibrium solution ¯x in a system of ordinary differential equations by assessing the sign of the real part of the eigenvalues of the Jacobian matrix evaluated at ¯x.[8]

The Cassini's Ovals, a geometric shape, are employed in conjunction with modified discs of Gershgorin to analyze the shape of a matrix and determine suitable limits for its roots. Additionally, we have demonstrated that by obtaining improved limits for polynomial coefficients with large values, Hessenberg matrices are recommended for estimating excellent bounds for the roots of the polynomial. However, these bounds tend to work better for smaller values. The primary objective of the effort was to identify a useful property of the Hessenberg form following the proposal of Dehmer's bound. Examples are provided to compare the obtained bounds with those obtained using traditional methods such as Cauchy's bounds, Montel's bounds, and Carmichel-Mason's bounds.

The large finite element global stiffness matrix represents a discrete, even-order differential operator with an algebraic form and zero row sums. However, Despite the theoretical assurance of positive definiteness, the direct application of Gershgorin's eigenvalue bounding theorem to this matrix consistently and reliably falls short in providing a significant lower bound for the smallest eigenvalue.

In order to address this, practical approaches are introduced for generating an ideal similarity transformation for the global stiffness matrix in real-world scenarios. By doing so, it becomes possible to uncover and improve nontrivial and practical lower bounds for the smallest eigenvalue. It is important to note that these approaches are applicable only to the common case where the global stiffness matrix exhibits negative off-diagonal values. The limitations of Gershgorin's theorem are taken into consideration when applying this technique to such matrices.

In exploring the parallel between eigenvalues and singular values, this review aims to compile various instances that highlight their similarities. One noteworthy illustration involves the resemblance between symmetric and rectangular Rayleigh quotients. Weyl's theorem, the Courant-Fischer minimax theorem, and their corollaries furnish fundamental properties of eigenvalues, while acknowledging the presence of "rectangular" variations of these theorems proves crucial. A fresh perspective on the subject emerges By comparing the characteristics of Rayleigh Quotient matrices and Orthogonal Quotient matrices, a unique solution emerges through the Orthogonal Quotients Equality. This transformation of Eckart-Young's minimal norm problem into an equivalent maximum norm problem reveals an unforeseen relationship between Ky Fan's maximum principle and the Eckart-Young theorem. It uncovers their interdependence as different aspects of the same underlying concept.

The Gershgorin theorem statement [9]

[8] Brûezin, E., & Zee, A. (1998). Non-Hermit Delocalization: Multiple Dispersion and Limits. Nuclear Physics B, 509, 599-614.

[9] Brouwer, P. W., Silvestrov, P. G., & Benaker, C. W. J. (1997). A Theory of Directed Localization in One Dimension. Physical Review B, 56, 4333-4335.

let **A** is a complex of n × n matrix with the entries in **C**.
The **A** eigenvalues belong to the Gershgorin disks union.
$\mathbf{D_i := \{z \in C : |z - a_{i,i}| \leq R_i \}}$.
$\mathbf{D_1, D_2, \ldots, D_n}$ are known as the **A** Gershgorin disks.
$\mathbf{R_i := \sum_{j \neq i} |a_{i,j}|}$
$\mathbf{R_i}$ is the absolute values sum of the of the non-diagonal entries in the rows **(i)** for $\mathbf{i \in \{ 1 , \ldots , n \}}$

Consider a matrix D = (aij) with elements aij, and let D(aij, R) ⊆ C represent a closed disc centered at aij with a known radius R. Each of these discs is referred to as a Gershgorin disc, it must lie within at least one of the Gershgorin discs, specifically within the disc D(aij, Ri), where Ri represents the radius of the Gershgorin disc centered at aij.

In accordance with the Gershgorin theorem, consider a complex matrix A of size n × n with entries a[i,j] for 1 ≤ i, j ≤ n. This theorem defines Gershgorin discs as follows:

- For each i = 1, 2, ..., n, compute Ri as the sum of the absolute values of the entries in the ith row of matrix A, excluding the diagonal element a[i,i].
- The Gershgorin disc Di represents the set of complex numbers z satisfying the inequality |z - a[i,i]| ≤ Ri, where i = 1, 2, ..., n.

The Gershgorin theorem asserts that every eigenvalue λi of matrix A must reside within at least one of the Gershgorin discs. In other words, the eigenvalues are guaranteed to be enclosed within the union of the Gershgorin discs.

The Gershgorin theorem Proof[10]

It’s a mathematical principle used to determine the bounds of eigenvalues in a square matrix. It was named after Semyon Aranovich Gershgorin, a Russian mathematician who introduced it in 1931. This theorem is particularly valuable when studying eigenvalues and analyzing the stability of linear systems.

every eigenvalue of a matrix A, represented as λ1, λ2, ..., λn, must reside within at least one Gershgorin disc. These discs are established using the matrix's entries, specifically by considering the sum of the absolute values of the elements in each row, except for the diagonal element. A Gershgorin disc, denoted as Di, encompasses a set of complex numbers z that adhere to the inequality |z - a[i,i]| ≤ Ri, where Ri signifies the aforementioned sum.

To provide a proof for the Gershgorin theorem, we consider an eigenvector x of matrix A that corresponds to an eigenvalue λi. Without loss of generality, let us assume that the ith component of x, denoted as x[i], is non-zero. By expanding the equation Ax = λx, we obtain a system of equations that encompasses each component of x.

By narrowing our attention to the ith equation, it becomes possible to rephrase it as λx[i] - (a[i,1]x[1] + a[i,2]x[2] + ... + a[i,n]x[n]) = 0. Upon taking the absolute value of both sides and applying the triangle inequality, we discover that |λ - a[i,i]| ≤ Ri, with Ri excluding the diagonal element.

This inequality shows that the eigenvalue λ lies within the Gershgorin disc Di. Since this inequality holds for every eigenvector x.

[10] van der Holst, H., Lovász, L., & Schrijver, A. (1999). Graph parameter Colin de Verdière. In Graph Theory and Combinatorial Biology, Mathematics of the Bolayi Society Study pp. 29-85.

To avoid the complexities, the simple method is done for registration of the medical image in which using the concept of the Gerschgorin circle theorem is proposed. newly, the Gerschgorin circle theorem has been utilized in the different science applications. thus, it has been aimed due to the mismatching or matching needing though the measurement between both the compared images inside the local distortions presence.[11]

For the example of The Gershgorin theorem[12]

Consider a 3x3 matrix A:

$$\mathbf{A} = \begin{vmatrix} 2 & 1 & 0 \\ 3 & -1 & 1 \\ -2 & 2 & 4 \end{vmatrix}$$

We first determine the Gershgorin discs before using the Gershgorin theorem. We calculate the total absolute value of each entry in each row i, omitting the diagonal element. Let's refer to these sums as R1, R2, and R3, respectively, for the first, second, and third rows.

- For the first row: R1 = |**1**| + |**0**| = **1**
- For the second row: R2 = |**3**| + |**1**| = **4**
- For the third row: R3 = |**-2**| + |**2**| = **4**

Now, we have the following Gershgorin discs:

D1: $|z - 2| \leq 1$ D2: $|z + 1| \leq 4$ D3: $|z - 4| \leq 4$

Next, we can find the eigenvalues of matrix A. Solving the characteristic equation det(A - λI) = 0, where I is the identity matrix, we get:

| 2-λ 1 0 | | 3 -1-λ 1 | = (2-λ)(-1-λ)(4-λ) + 3(4-λ) + 3 = 0 | -2 2 4-λ |

Expanding and simplifying the determinant equation, we have:

[11] Chaitin-Chaitelin, F., & Harrabi, A. (2015). On Definitions of Pseudo-spectrums of closed operators in Banach spaces pp 120-132

[12] "Favard's Adequacy: A condition for a class of axis-controlled operators" authored by S. N. Chandler-Wilde and M. Lindner. The paper was published in the Journal of Functional Analysis, volume 254, and spans pages 1146-1159 in the year 2008..

$(2-\lambda)(-1-\lambda)(4-\lambda) + 3(4-\lambda) + 3 = 0$ $(\lambda-2)(\lambda+1)(\lambda-4) + 3(4-\lambda) + 3 = 0$ $(\lambda-2)(\lambda+1)(\lambda-4) - 3(\lambda-4) - 3 = 0$ $(\lambda-2)(\lambda+1)(\lambda-4) - 3\lambda + 12 - 3 = 0$ $(\lambda-2)(\lambda+1)(\lambda-4) - 3\lambda + 9 = 0$

Solving this equation, we find three eigenvalues:

$\lambda 1 = -1$ $\lambda 2 = 3$ $\lambda 3 = 6$

Now, let's check if these eigenvalues lie within the Gershgorin discs:

- For $\lambda 1 = -1$: -1 lies within **D2: $|z + 1| \leq 4$**
- For $\lambda 2 = 3$: 3 lies within **D1: $|z - 2| \leq 1$ 3 lies within D3: $|z - 4| \leq 4$**
- For $\lambda 3 = 6$: 6 lies within **D3: $|z - 4| \leq 4$**

As we can see, all the eigenvalues are indeed contained within at least one of the Gershgorin discs, confirming the validity of the Gershgorin theorem for this matrix.

The Gershgorin theorems
They can be divided into[13]:

The Gershgorin Theorem 1: [14]

Gershgorin Theorem in Respect to the columns in which that (Every matrix eigenvalue should lie in a Gershgorin disc that corresponds to the **A** column.

The Gershgorin Theorem 1 Proof
According to the Gershgorin theorem, These discs are established using the entries of matrix A, wherein each disc possesses a radius equivalent to the sum of the absolute values of the remaining elements in the corresponding row. Additionally, each disc is centered around the diagonal element a[i,i].

The Gershgorin Theorem 2 [15]

Gershgorin's theorem is presented in two versions: Gershgorin's theorem 1, which offers a straightforward method to approximate the eigenvalues of a matrix, and Gershgorin's theorem 2, which provides more accurate bounds on the eigenvalue locations.

Gershgorin's theorem 2 provides a refined estimation of the eigenvalue positions compared to Gershgorin's theorem 1. It applies to an n x n matrix A = [a_ij], where R_i represents the sum of the absolute

[13] Horn, R. A., & Johnson, C. R. (2013). Matrix Analysis (2nd ed.). Cambridge University Press.

[14] Li, C.-K., & Zhang, F. (2019). Eigenvalue continuity and Gersgorin theorem. Electronic Journal of Linear Algebra (ELA), 35, 619–625.

[15] Gershgorin, S. (1931). Über die Abgrenzung der Eigenwerte einer Matrix. Izv. Akkad. Knock. USSR Otd. Fizz is dead. Knock (in German), 6, 749-754

values of the off-diagonal entries in the i-th row of A, denoted as R_i = sum_{j != i} |a_ij|, for i = 1, 2, ..., n.

According to Gershgorin's theorem 2, any eigenvalue λ of matrix A must satisfy at least one of the inequalities: |λ - a_ii| ≤ R_i, where i ranges from 1 to n. This implies that the eigenvalues of A are constrained to lie within the union of n disks in the complex plane. Each disk is centered at a_i and has a radius of R_i. By using this theorem, we can obtain a more precise estimation of the potential locations of the eigenvalues compared to Gershgorin's theorem 1.

The Gershgorin Theorem 2 Proof[16]

Gershgorin discs with a radius of a, with the eigenvalues and diagonal elements centred at aa, and the eigenvalues being equal to the diagonal elements.[17]

Along with the eigenvalues moving, the Gershgorin discs of A (p), which are based on p, will likewise grow in radius as p grows. The various polynomial roots of the A'(p), where p is specified from 0 to 1, can be used to trace the motions.[18]

The continuous distinct polynomial must range from 0 to 1.by assuming that A An. and that A'(p) is the matrix A that has the diagonal components multiplied by the p variables.

A'(p)'s characteristic polynomial is seen as a function depending on the variables p and x. between these parameters The roots are regarded as continuous if the distinct polynomial is one that is.

The eigenvalue changes that occur as p 1 are continuous because the eigenvalues are discovered with the roots. Therefore, based on the p variable, the eigenvalues must follow a continuous route from one location to the next.[19]

The Gershgorin Theorem 3 (disjoint Gershgorin Disc)

This theorem applies to the stability of dynamic systems. If a real diagonal element with a row can be used to form matrix An, which includes the disjoint Gershgorin Disc, P, then the eigenvalue inside the disc P is rea.[20]

The Gershgorin Theorem 3 Proof

[16] Brualdi, R. A., & Mellendorf, S. (1994). Regions in the complex plane containing the eigenvalues of a matrix. American Mathematics Monthly, 101, 975-985.

[17] MacCluer, C. R. (2000). Several proofs and applications of Perron's theory. SIAM Review, 42, 487-498.

[18] Scott, D. S. (1985). On the accuracy of Gershgorin's circle theory for constraining real symmetric matrix diffusion. Linear Algebra and Its Applications, 65, 147-155.

[19] eingold, D. G., & Varga, R. S. (1962). Block Diagonally Dominant Matrices and the Publications of the Gerschgorin Circle Theorem. Pacific Journal of Mathematics, 12, 1241-1250.

[20] Gradshteyn, I. S., & Ryzhik, I. M. (2000). Tables of Integrals, Series, and Products (6th ed.). San Diego, CA: Academic Press, pp. 1073-1074..

By assuming that A belongs to the set An, we can conclude that λ is an eigenvalue of An, which falls within a disc formed by one of the rows of A, where a real diagonal element exists. [21]

When examining an eigenvalue λ of matrix An, where λ = x + iy and x and y are non-zero real numbers, it can be deduced that the complex conjugate of λ, denoted as λ', is also an eigenvalue of An. The complex conjugate of a complex number is obtained by changing the sign of its imaginary part. Therefore, if λ = x + iy, then λ' = x - iy.

Due to the equidistant nature of x-iy from the center of the disc, which corresponds to x+iy, it can be concluded that x-iy lies within the disc denoted as q. Consequently, two eigenvalues exist within the isolated Gershgorin Disc q, positioned at the center of the real axis. [22]

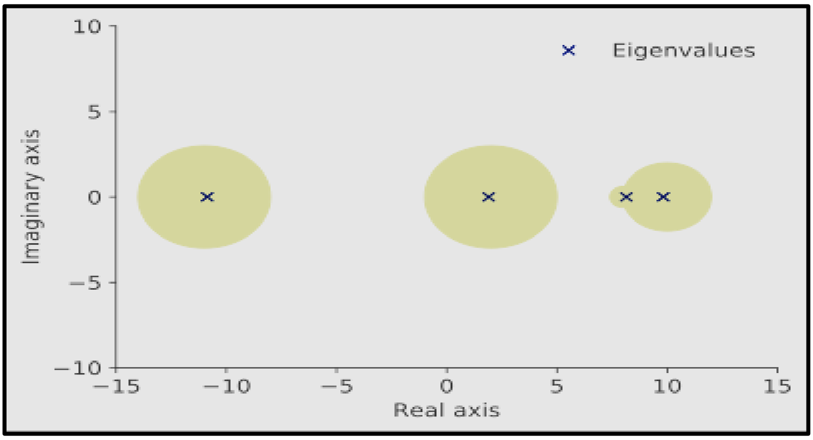

Figure 1: The Gershgorin theorem for estimating eigenvalues

Figure 1 showcases the application of the Gershgorin theorem for estimating eigenvalues. The figure visually represents the eigenvalue-associated discs, which are highlighted in yellow. It is visually apparent that the first two discs overlap, resulting in the containment of two eigenvalues within their combined area.

Iterative Methods

Stationary Method:

[21] Piziak, R., & Turner, D. (1994). Investigating Gerschgorin Circles and Cassini Ovals. Journal of Mathematics Education and Research, 3, 13-21.

[22] Scott, D. S. (1985). The Accuracy of the Gerschgorin Circle Theorem in Estimating the Range of a Real Symmetric Matrix. Linear Algebra and Its Applications, 65, 147-155.

The stationary iterative method is characterized by the equation x(k+1) = Gx(k) + c, where G and c are constant values that do not vary with the iteration count (k).

- **Non-Stationary Method:**

The non-stationary iterative method is defined by the equation x(k+1) = x(k) + akp(k), where the computation involves information that changes at each iteration.

- **Stationary Method: Jacobi Method**

Within the realm of stationary iterative methods, the Jacobi method is dedicated to solving for the value of xi in the i-th equation, while presuming the remaining entries of vector x to be fixed. This method can be represented in matrix notation as: x = (D^(-1))(b - Lx - Ux) In the aforementioned equation, D symbolizes the diagonal matrix, L represents the strictly lower triangular matrix, U denotes the strictly upper triangular matrix, x corresponds to the solution vector, and b signifies the right-side vector.

Application of The Gershgorin theorem[23]

When confronted with matrix equations of the form Axe = b, where b represents a vector and A represents a matrix with a high condition number, the Gershgorin circle theorem proves to be a valuable tool. In such scenarios, the resulting error of the solution often corresponds to the initial error of the data multiplied by the condition number of A. To provide an example, if the condition number of A exceeds 1000, it becomes necessary to know the vector b up to six decimal places in order to obtain a solution x that is accurate to three decimal places. This example illustrates the significant influence of the condition number of A on the potential precision of the final result.

Rounding mistakes of any size can have a significant impact on the outcome. becomes unreliable, especially for very high condition numbers. To mitigate this issue, one approach is to precondition the equation as PAx = Pb, where P is a matrix satisfying P A ≈ I. Computing the precise inverse of A can be computationally expensive, but by using an approximate inverse, the preconditioning can be accomplished.[24]

Discussion of The Gershgorin theorem

Examining a square matrix made up of tiny magnitude off-diagonal entries in the complex number field is one method for proving the theorem. It is vital to understand that the matrix's eigenvalues cannot considerably depart from the diagonal entries. The magnitudes of the off-diagonal entries can be decreased, making it possible to satisfy this condition. A rough approximation of the matrix's eigenvalues is possible

[23] Taussky-Todd, O. (1949). A Recurring Theorem on Determinants. American Mathematical Monthly, 56, 672-676.

[24] Varga, R. S. (2004). Geršgorin and His Circles. Berlin: Springer-Verlag..

because to this change in the off-diagonal elements. As a result, despite minimising the off-diagonal entries, the diagonal elements may change in several ways.[25]

If the discs are associated with the axes in Cn, and each disc represents a limit on the eigenvalues that correspond to Eigen spaces closest to a specific axis in the matrix, then the theorem does not guarantee the existence of one disc for each eigenvalue.[26]

By the eigenvalues **a, b** and **c** construction that has the eigenvectors to identify the disc for 2 rows **a** and **b** while the disc that contains of 3 rows **a**, **b** and **c**.

The theorem ensures that, when taking into account all three eigenvalues, the radius of a single disc can produce an estimate that is double the sum of the radii of the other two discs.[27]

The test is used to establish the local stability requirements for dynamical system equilibrium solutions. The criteria is highly useful since it enables us to create stability requirements without having to figure out the eigenvalues of the dynamic system's Jacobian matrix.

The qualitative analysis of solutions within dynamical systems, as defined by ordinary differential equations, plays a crucial role in understanding biological phenomena within various domains of applied mathematics, including biomathematics. For researchers engaged in the qualitative examination of dynamic systems, this test proves highly valuable and beneficial in facilitating their investigation.

Strengthening of The Gershgorin theorem[28-29]

The strengthening of Gershgorin's theorem that you mentioned is a useful observation that can help to determine the number of eigenvalues in each disk. The theorem states that if a disk can be separated from the others, then it must contain precisely one eigenvalue. If it encounters another disk, then it might not contain any eigenvalues. [30]

for example,

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[25] Anderson, G., & Zeitouni, O. (2008). A Law of Large Numbers for Finite-Range Dependent Random Matrices. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 61(8), 1118–1154.

[26] Anderson, G. W., Guionnet, A., & Zeitouni, O. (2018). An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics, Vol. 118. Cambridge University Press.

[27] Anshelevic, M. (2009). Appell polynomials and their relatives II. Boolean theory. Indiana Univ. Math. J., 58(2), 929–968.

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$$A=\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix}$$

Let's consider the matrix A = [0 1; 0 4] as an example. We can apply Gershgorin's theorem to obtain an estimate of the location of the eigenvalues of A.

First, we compute the sums R_i for each row of A:

R_1 = 1

R_2 = 0

In this case, we have:

- a_1 = 0, with disk centered at 0 and radius 1
- a_2 = 4, with disk centered at 4 and radius 0

The disks are shown in Figure 1 as shaded regions.

From the theorem, we know that the disk centered at 4 must contain precisely one eigenvalue, since it is separated from the other disk. On the other hand, the disk centered at 0 might not contain any eigenvalues, since it encounters the other disk.

The (PROOF) [31-32]

Let's continue by considering matrix D as a diagonal matrix whose elements are identical to those of matrix A. We will proceed with the expression B(t) = (1-t)D + tA.

It can utilise the continuous eigen values to demonstrate whether any eigen values shift from one union to the other, in which case they should be outside of each disc.

assertion is true for D= b(0).[33]

The Gershgorin circles' centres are identical since b(t)'s diagonal entries are equal to those of A's, but their radii are t times larger. As a result, the remaining n-k union and the accompanying k discs union are no longer connected. for all t ε (0,1).

With the assumption of closed discs, the designated distance between the two unions for matrix A is denoted as D0. The distance b(t) is consistently greater than or equal to D, as it is regarded as a decreasing function of t.

Moreover, due to the continuous nature of the eigenvalues of b(t), the distance of any λ(t) eigenvalue within the union of k discs from the remaining n - k discs is also considered as continuous.

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[31] N. Chandler−Wilde and M. Lindner. This paper is published as a Memoir of the American Mathematical Society (AMS) with the identifier TU Chemnitz−7, NI08017−HOP of INI Cambridge.

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There are two forms of eigenvalues-related continuity:[34-]

1- Each unique eigenvalue is considered as a conventional continuous function, limited to the real interval as the only feasible range of occurrence.
2- By employing a norm-induced metric, the eigenvalues are transformed from the matrix space into unordered tuples. In this context, they are perceived as a continuous collection in a topological manner. This transformation specifically occurs within the nC quotient space, maintaining permutation equivalence. The utilization of the continuity of this mapping justifies the application of the Gershgorin disc theorem..[35]

No eigenvalue continuity of any type is needed for the complicated analysis argument principle demonstration.

Utilizing a unique approach, the application of the Gershgorin circle theorem has revealed a new method for detecting hierarchical clusters in a provided dataset. By implementing the theorem specifically on the normalized Laplacian matrix, one can derive upper bounds for the eigenvalues. This crucial information aids in the identification and comprehension of the hierarchical framework inherent within the dataset.[36-37]

In order to streamline the process of identifying the optimal value of k at each level, it was essential to obtain the relevant intervals. To facilitate the acquisition of eigenvalues and eigenvectors, conducting computationally intensive operations on eigenvalues proves advantageous. By utilizing the normalized Laplacian matrix, any spectral clustering technique can incorporate the provided intervals for k as input. The combined implementation of this method with spectral clustering showcases its efficacy in generating hierarchical clusters for diverse synthetic and real-world datasets.[38-39]

Conclusion

The Gershgorin theorem is usually stated with a part in the supplementary state to give the most information about the eigen values location in which the discs union to form all the union of connected-component after that the eigen values lie exactly including that.

For symmetric finite global element matrices with non-positive diagonal members and positive definiteness, this was regarded as the known state. Therefore, with a smaller least eigen value, the finite global element stiffness matrix easily grows to be quite huge.

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It has been demonstrated how the matrix's best similarity transformations only need matrix-vector multiplication, which results in a feasible and reliable set of numerical methods for the tight realistic alternative.

the global stiffness matrices' finite big elements' least eigenvalue's Gershgorin limits.

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