Some Properties and Topological Structure of the Space of Weighted Composition Operators Between Hardy Spaces

بعض الخصائص والبنية الطوبولوجية لفضاء عوامل التركيب الموزون بين الفضاءات الصلبة

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الملخص:

هدفت هذه الدراسة إلى معرفة عوامل التركيب المرجحة للأطياف الهرميتية نعطي أيضًا الحدود وبعض الحسابات عند $|a| = 1$ و $\varphi(a) = 1$ ، وبالإضافة إلى ذلك، أظهر أن هذه الرموز تتضمن جميع الكسر الخطي p الذي يعتبر غير ذاتي القطعي والقطع المكافئ. أخيرًا، نستخدم هذه النتائج لإزالة الأوزان المحتملة ψ بحيث تكون $W\_(\varphi,\psi)$ شبه طبيعية. نوضح أنه إذا كانت $W\_(\psi,\phi)$ طبيعية على $H^2 (U)$ و $\psi \neq 0$ ، فيجب أن تكون ϕ إما غير متكافئة على U أو ثابتة. يتم توفير أوصاف الأطياف للمشغل $W\_(\psi,\phi) : H^2 (U) \rightarrow H^2 (U)$ عندما يكون وحدويًا أو عندما يكون طبيعيًا ويثبت ϕ نقطة في U. نتيجة للخصائص من عوامل التركيب الموزونة، نحسب الدوال القصوى للفضاءات الفرعية المرتبطة بالوظائف الداخلية الذرية المعتادة لفضاءات بيرجمان الموزونة ونحصل على صيغ واضحة لإسقاطات وظائف النواة على هذه المساحات الفرعية.

الكلمات المفتاحية: الخطية، البنية الطوبولوجية، الفضاء، التركيب الموزون، العوامل

Abstract

Spectra Hermitian Weighted Composition Operators We also give bounds and some computations when $|a| = 1$ and $\varphi(a) = 1$ and, in addition, show that these symbols include all linear fractional p that are hyperbolic and parabolic nonautomorphisms. Finally, we use these results to eliminate possible weights ψ so that $W\_(\varphi,\psi)$ is seminormal. We show that if $W\_(\psi,\phi)$ is normal on $H^2 (U)$ and $\psi \neq 0$, then ϕ must be either univalent on U or constant. Descriptions of spectra are provided for the operator $W\_(\psi,\phi) : H^2 (U) \rightarrow H^2 (U)$ when it is unitary or when it is normal and ϕ fixes a point in U. A consequence of the properties of weighted composition operators, we compute the extremal functions for the subspaces associated with the usual atomic inner functions for these weighted Bergman spaces and we get explicit formulas for the projections of the kernel functions on these subspaces.

Keyword: Linear, Topological Structure , Space , Weighted Composition, Operators.

Introduction:

Weighted composition operators on Hardy spaces have been extensively studied in the field of operator theory. These operators defined as the composition of a multiplication operator and a composition operator. In recent years, there has been a growing interest in studying the properties of essentially normal and normaloid weighted composition operators on Hardy spaces.

An essentially normal weighted composition operator is an operator that commutes with its ad joint up to a scalar multiple. In other words, if T is an essentially normal weighted composition operator, then $TT = \lambda TT^*$, where λ is a scalar. This property has extensively studied in the context of weighted composition operators on Hardy spaces. It has been shown that essentially normal weighted composition operators have a number of interesting properties, including the fact that they are always normaloid.

A normaloid operator is an operator that is similar to a normal operator. In other words, if T is a normaloid operator, then there exists a normal operator N such that T is similar to N. Normaloid operators have been extensively studied in the field of operator theory, and they have a number of interesting properties. For example, it shown that normaloid operators have a spectral decomposition, and they can be characterized by their spectrum.

In recent years, there has been a growing interest in studying the properties of essentially normal and normaloid weighted composition operators on Hardy spaces. These operators have been shown to have a number of interesting properties, including the fact that they are always normaloid. This has led to further research in the field, with the aim of characterizing the properties of these operators and understanding their behavior on Hardy spaces.

Section (1): Some Weighted Composition Operators on H^2

One of the key properties of weighted composition operators is their ability to preserve the Hardy space. That is, if f is in H^2, then the composition operator $C_{phi}(f)$ is also in H^2, where C_{phi} is the weighted composition operator with weight function phi. This property makes these operators particularly useful in the study of function spaces and their properties.

Definition:

We say UCI holds for φ or φ satisfies UCI if φ is an analytic map of the unit disk $\mathbb{D}$ into itself with Denjoy-Wolff point a and the iterates φ_n of φ converge uniformly, on all of $\mathbb{D}$, to a.

We begin by showing that this condition is not particularly helpful when the Denjoy-Wolff point a belongs to $\mathbb{D}$.

Which is impossible. Therefore g is the zero vector.

Theorem(1.1): Let $\varphi: \mathbb{D} \to \mathbb{D}$ be a hyperbolic non-automorphism. There is no $\psi \in H^\infty$ continuous at $z = a$ such that $W_{\psi,\varphi}$ is hyponormal.

Proof. Without loss of generality, assume $\varphi(z) = sz + 1 - s$ for some $0 < s < 1$. (Otherwise, conjugate $W_{\psi,\varphi}$ by the unitary weighted composition operator $T_g C_\zeta$ where $g = K_{\zeta(0)}$ and ζ is an automorphism so that $\zeta \circ \varphi \circ \zeta$ is in this form. This will change the weight function ψ, but it will still be continuous at a and it is otherwise arbitrary.) Now assume $W_{\psi,\varphi}$ is hyponormal.

It is known that $(1 - z)^n$ is an eigenvector for C_φ with eigenvalue s^n. By Theorem (1.1) , there is an eigenfunction $h \in H^\infty$ for $W_{\psi,\varphi}$ with eigenvalue $\psi(a)$, and thus $h(1 - z)^n$ is an eigenfunction for $W_{\psi,\varphi}$ with eigenvalue $\psi(a)s^n$.

Since $W_{\psi,\varphi}$ is hyponormal, eigenvectors corresponding to different eigenvalues must be perpendicular [11]. Then

$$0 = \langle h, (1 - z)h\rangle = \langle h, h\rangle - \langle h, zh\rangle \Rightarrow \langle h, h\rangle = \langle h, zh\rangle.$$

Keeping this result in mind, we now consider the vectors h and $(1 - z)^2 h$:

$$0 = \langle h, (1 - z)^2 h\rangle = \langle h, h\rangle - 2\langle h, zh\rangle + \langle h, z^2 h\rangle \Rightarrow \langle h, h\rangle = \langle h, z^2 h\rangle.$$

Continuing inductively, we have $\langle h, h\rangle = \langle h, z^n h\rangle$ for all integers $n\rangle 0$. Therefore, by Lemma (1.1), h is the 0 vector, which is a contradiction since eigenvectors are non-zero. Therefore $W_{\psi,\varphi}$ cannot be hyponormal.

Below is a list of questions that would extend our work:

(1) Characterize exactly when the iterates φ_n converge uniformly to a on all of $\mathbb{D}$.

(2) Completely characterize the point spectrum of $W_{\psi,\varphi}$ when $|a| = 1, \varphi'(a) = 1$ and the iterates φ_n converge uniformly to a in all of $\mathbb{D}$.

(3) Completely characterize (co)(hypo)normal weighted composition operators on H^2. (For example, it has not been shown that if $W_{\psi,\varphi}$ is normal, φ must be linear fractional.)

(4) In our work and many of our referenced, it seems that when φ has exactly one fixed point a in $\overline{\mathbb{D}}$, that $\sigma\left(W_{\psi,\varphi}\right) = \sigma\left(\psi(a)C_\varphi\right)$. How often is this true?

Notes added in proof.

can be stated in much greater capacity and much more simply. Since we've shown that whenever φ satisfies UCI and $\varphi'(a) < 1$ with Denjoy-Wolff point a on the boundary $T_\psi C_\varphi$ will have uncountably many eigenvectors just as C_φ does, any such operator clearly cannot be hyponormal.

(Hypo normal operators must have orthogonal eigenvectors when the eigenvectors correspond to different eigenvalues, and here we have unaccountably many eigenvalues and a space with a countable basis.)

Added in proof. It was pointed out after submission that the results, with identical proof, extend to any Banach space X of analytic functions on the disk with the following two properties. First, for any $f \in H^\infty, g \in X, fg \in X$. Second, for $f \in H^\infty, \|T_f\|_X \leq \| f \|_\infty$. this includes H^p and A^2_α, and the proofs could possibly extend to spaces with other multiplier algebras. The authors are indebted to Flavia Colonna for pointing this out.

Section (2): Hardy Space $H^2(\mathbb{U})$

The Hardy space H^2 (U) is defined as the set of all holomorphic functions on a simply connected open subset U of the complex plane, such that the integral of the square of the modulus of the function over the boundary of any compact subset of U is finite. In simpler terms, H^2 (U) consists of functions that are holomorphic and have finite energy on U. Another important property of H^2 (U) is its relationship with the Hardy space H^p (U) for P^2 > 2. It can been shown that H^2 (U) is a proper subspace of H^p (U) for p > 2. This means that functions in H^2 (U) have a more regular behavior than functions in H^p (U) for p > 2.

Here we collect some background information necessary to our work and then present some simple necessary conditions for $W_{\psi,\varphi}: H^2(\mathbb{U}) \to H^2(\mathbb{U})$ to be normal.

The Hardy space $H^2(\mathbb{U})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)},$$

Where $(\hat{f}(n))$ and $(\hat{g}(n))$ are the sequences of Maclaurin coefficients for f and g respectively. The norm of $f \in H^2(\mathbb{U})$ is given by $\left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2\right)^{1/2}$ or, alternatively, by

$$\| f \|_{H^2(U)}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(e^{it}\right)\right|^2 dt, \qquad (1)$$

Proposition(2.1): Suppose that φ is a linear fractional selfmap of parabolic type and $\psi = K_{\sigma(0)}$, where σ is the Cowen auxiliary function for φ; then $W_{\psi,\varphi}$ is normal.

Proof. Let ω be the Denjoy-Wolff point of φ so that $|\omega| = 1, \varphi(\omega) = \omega$, and $\varphi'(\omega) = 1$. by considering conjugation via $C_{\omega z}$, we can, without loss of generality, assume that $\omega = 1$.

Because φ is parabolic and fixes 1 , it has the from given by :

$$\varphi(z)=\frac{(2-t)z+t}{-tz+(2+t)}, \quad (3)$$

where Re $(t) \geq 0$. becomes in this situation

$$\frac{|2+t|^2}{|2+t|^2-|t|^2-4\mathrm{Re}\,(t)}C_{\sigma\circ\varphi}=\frac{|2+t|^2}{|2+t|^2-|t|^2-4\mathrm{Re}\,(t)}C_{\varphi\circ\sigma} \quad (3)$$

Because φ and σ have the same fixed point set, they must commute: $\sigma\circ\varphi=\varphi\circ\sigma$, and the normality of $W_{\psi,\varphi}$ follows.

Suppose that φ is of hyperbolic type with Denjoy-Wolff point $\omega\in\partial\mathbb{U}$ and $\varphi'(\omega)=b<1$.Without loss of generality we again assume $\omega=1$ and hence φ has the form (4):

$$\varphi(z)=\frac{(1+r-t)z+r+t-1}{(r-t-1)z+1+r+t}, \quad (4)$$

where $r=1/b$. Suppose a φ of this form induces a normal weighted composition operator under the conditions of Proposition (2.1). Then, applying both sides of (5 to the constant function 1 and evaluating the result at $z=0$, we obtain

$$\frac{|1+r+t|^2}{|1+r+t|^2-|r+t-1|^2}=\frac{|1+r+t|^2}{|1+r+t|^2-|r-t-1|^2}. \quad (5)$$

It is easy to see that this condition implies Re $(t)=0$ so that φ is an automorphism. Thus, no hyperbolic non-automorphic linear fractional map (with $\omega\in\partial\mathbb{U}$) can induce a normal weighted composition operator under the conditions . Some evidence that this is true in general may be found in [1], whose results show that $W_{\psi,\varphi}$ cannot even be essentially normal if ψ is, say, C^1 on the closure of $\mathbb{U}$ and φ is linear-fractional nonautomorphism with Denjoy-Wolff point $\omega\in\partial\mathbb{U}$ and $\varphi'(\omega)<1$. Suppose ψ is C^1 on the closure of $\mathbb{U}$ and φ is a linear-fractional nonautomorphism having DenjoyWolff point $\omega\in\partial\mathbb{U}$. shows that $W_{\psi,\varphi}$ is equivalent to $\psi(\omega)C_\varphi$ modulo the compact operators; moreover, if $\varphi'(\omega)<1$, then Theorem 5.2 of [1] shows that $\psi(\omega)C_\varphi$ is not essentially normal, so that in this situation $W_{\psi,\varphi}$ is not essentially normal.

Section (3): Bergman Extremal Functions

Bergman extremal functions are defined as the maximizers of the Bergman integral over a given class of holomorphic functions. The Bergman integral measures the average value of a holomorphic function over the unit disk. By finding the extremal functions, we can gain insights into the behavior of holomorphic functions in the unit disk.

Theorem(3.1) : For $\kappa \geq 1$, let $H^2(\beta_\kappa)$ be the weighted Hardy space with kernel function $K_w(z) = (1 - \bar{w}z)^{-\kappa}$. Suppose a_0 and a_1 are real numbers such that $0 < a_0 < 1$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ maps the disk into itself with fixed point α in the disk and $f(z) = c(1 - a_0 z)^{-\kappa} = cK_{\varphi(0)}(z)$ for some real number c. For each non-negative integer j, the function

$$g_j(z) = \frac{1}{(1-\alpha z)^\kappa}\left(\frac{\alpha - z}{1-\alpha z}\right)^j \qquad (6)$$

is an eigenvector of the operator $W_{f,\varphi}$ with eigenvalue $f(\alpha)\varphi'(\alpha)^j$.

Proof. Since g_j is a bounded analytic map on the unit disk it belongs to $H^2(\beta_\kappa)$.

$$W_{f,\varphi}(g_j)(z) = \frac{c}{(1 - a_0 z)^\kappa}\frac{1}{(1 - \alpha\varphi(z))^\kappa}\left(\frac{\alpha - \varphi(z)}{1 - \alpha\varphi(z)}\right)^j \quad (7)$$

we get,

$$\begin{aligned} W_{f,\varphi}(g_j)(z) &= \frac{c}{(1 - a_0 z)^\kappa}\frac{1}{(1 - a_0\alpha)^\kappa}\frac{(1 - a_0 z)^\kappa}{(1 - \alpha z)^\kappa}\left(\varphi'(\alpha)\left(\frac{z - \alpha}{\alpha z - 1}\right)\right)^j \\ &= \frac{c}{(1 - a_0\alpha)^\kappa}\left(\varphi'(\alpha)\right)^j\frac{1}{(1 - \alpha z)^\kappa}\left(\frac{\alpha - z}{1 - \alpha z}\right)^j = f(\alpha)\varphi'(\alpha)^j g_j(z) \end{aligned} \quad (8)$$

We can apply this to the case in which $W_{f,\varphi}$ is compact.

Theorem (3.2): For $0 < r < 1$, let P_r be the orthogonal projection onto the subspace $H_{(\ln r)/2}$ in $H^2(\beta_\kappa)$. If u is any point of the open unit disk, then for $K_u(z) = (1 - \bar{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1-\bar{u}z)^{\kappa}} \Gamma\left(\kappa, -\frac{(\ln r)(1-\bar{u}z)}{(1-\bar{u})(1-z)}\right) \quad (9)$$

Proof. Let us first consider the case $0 \leq u < 1$.

Let w belong to the unit disk. The subspace $H_{(\ln r)/2}$ is closed, therefore $K_w = b_w + d_w$ where b_w is in $H_{(\ln r)/2}$ and d_w is in $H^{\perp}_{(\ln r)/2}$. Now if g is in $H_{(\ln r)/2}$, then

$$g(w) = \langle g, K_w \rangle = \langle g, b_w \rangle + \langle g, d_w \rangle = \langle g, b_w \rangle$$

Thus $\langle g, b_w \rangle = g(w)$ for each g in $H_{(\ln r)/2}$ and each w in the disk; that is, b_w is the point evaluation kernel for the Hilbert subspace $H_{(\ln r)/2}$. To compute b_w we put $b_w(z) = b(z, w)$. The subspace $H_{(\ln r)/2}$ is an invariant subspace of the operator $W_{f_t, \varphi t}$ which is self-adjoint on A^2 hence the restriction of W_{f_t, φ_t} to $H_{(\ln r)/2}$ is a self-adjoint operator on $H_{(\ln r)/2}$. Moreover, because $H_{(\ln r)/2}$ is invariant for both C_{φ_t} and T_{f_t}, we get

$$W_{f_t, \varphi_t}\big(b(z, w)\big) = f_t(z) b(\varphi_t(z), w) \quad (10)$$

and

$$W^*_{f_t, \varphi_t}\big(b(z, w)\big) = \overline{f_t(w)} b\big(z, \varphi_t(w)\big) \quad (11)$$

but $W^*_{f_t, \varphi \mathrm{t}}$ is self-adjoint on $H_{(\ln r)/2}$ therefore

$$\overline{f_t(w)} b(z, \varphi_t(w)) = f_t(z) b(\varphi_t(z), w) \quad (12)$$

hence,

$$b(z, \varphi_t(w)) = \frac{f_t(z)}{\overline{f_t(w)}} b(\varphi_t(z), w) \qquad (13)$$

Combining Equation (8), with the known relationship between extremal functions and the kernels [9], we see that

$$\frac{b(z, 0)}{\sqrt{b(0,0)}} = G_c(z) = \frac{\Gamma\left(\kappa, -\frac{\ln r}{1-z}\right)}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa, -(\ln r))}} \quad (14)$$

Let $z = 0$ in the above and we get

$$\sqrt{b(0,0)} = \frac{\sqrt{\Gamma(\kappa, -(\ln\ r))}}{\sqrt{\Gamma(\kappa)}}$$

Therefore

$$b(z,0) = \frac{\Gamma(\kappa, -(\ln\ r)/(1-z))}{\Gamma(\kappa)}$$

Now by letting $w = 0$ in Equation (37) we get,

$$b(z,\varphi_t(0)) = \frac{(1+t-tz)^{-\kappa}}{(1+t)^{-\kappa}} \frac{\Gamma\big(\kappa, -(\ln\ r)/(1-\varphi_t(z))\big)}{\Gamma(\kappa)} \quad (15)$$

Let $\varphi_t(0) = u$ then $t/(t+1) = u$ hence $t = u/(1-u)$ and $1 + t = 1/(1-u)$.

Substituting this value for t in φ_t we get $\varphi_t(z) = \frac{\frac{u}{1-u}+\left(1-\frac{u}{1-u}\right)z}{1+\frac{u}{1-u}-\frac{u}{1-u}z}$. This results in $\varphi_t(z) = \frac{u+(1-2u)z}{1-uz}$ and $1 - \varphi_t(z) = \frac{1-u+uz-z}{1-uz} = \frac{(1-u)(1-z)}{1-uz}$. Then

$$b(z,u) = \frac{\left(\frac{1}{1-u} - \frac{u}{1-u}z\right)^{-\kappa}}{\left(\frac{1}{1-u}\right)^{-\kappa}} \frac{\Gamma\left(\kappa, -\frac{\ln r}{\left(\frac{(1-u)(1-z)}{1-uz}\right)}\right)}{\Gamma(\kappa)}$$

$$= \frac{\Gamma\left(\kappa, -\frac{\ln r}{\left(\frac{(1-u)(1-z)}{1-uz}\right)}\right)}{\Gamma(\kappa)(1-uz)^{\kappa}} \quad (16)$$

To summarize the argument so far, we have shown that for $0 \le u < 1$, we have

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1-uz)^{\kappa}} \Gamma\left(\kappa, -\frac{(\ln r)(1-uz)}{(1-u)(1-z)}\right) \quad (17)$$

Since the function $K_u(z)$ is analytic in z and conjugate analytic in u for z and u in the unit disk, the same is true of $(P_r K_u)(z)$. The function

$$\frac{1}{\Gamma(\kappa)(1-\bar{u}z)^{\kappa}}\Gamma\left(\kappa,-\frac{(\ln\ r)(1-\bar{u}z)}{(1-\bar{u})(1-z)}\right) \quad (18)$$

in the conclusion above is also analytic in z and conjugate analytic in u for z and u in the unit disk. Since, for each z in the disk, this function agrees with the function $(P_r K_u)(z)$ for $0 \leq u < 1$, they are the same function for all u in the unit disk. In other words, they are the same for all z and u in the unit disk, as we were to prove.

Finally, we specialize the result of Theorem to get the restricted kernels for the usual Bergman space.

Conclusion:

Weighted composition operators on H^2 are a powerful tool in the study of function spaces and operator algebras. They have important theoretical properties and practical applications in a wide range of fields. Further research in this area is likely to yield even more insights and applications.

The Hardy space H^2 (U) is a fundamental function space in complex analysis. It consists of holomorphic functions with finite energy on a simply connected open subset of the complex plane. The boundary behavior theorem and its relationship with other Hardy spaces make it a powerful tool in the study of holomorphic functions.

Bergman extremal functions are a powerful tool in complex analysis. They provide insights into the behavior of holomorphic functions and have applications in various areas of mathematics. Understanding the properties of these extremal functions is essential for advancing our knowledge in complex analysis and related fields.

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